Abstraction in Context, Combining Constructions, Justification and Enlightenment

Ivy Kidron
Jerusalem College of Technology, Israel
<ivy@jct.ac.il>

Tommy Dreyfus
Tel Aviv University, Israel
<tommyd@post.tau.ac.il>

The nested epistemic actions model of abstraction in context has been used to analyse a solitary learner's process of justification. In previous work, we have shown that this process gave rise to the phenomenon of parallel interacting constructing actions. In this paper, we analyse the interaction pattern of combining constructions, and show that combining constructions indicate an enlightenment of the learner. This adds an analytic dimension to the nested epistemic actions model of abstraction in context.

Introduction and Background

Abstraction has been a central issue in mathematics and science education for many years. The classic work by Piaget, Davydov, Skemp and others has in recent years been succeeded by research fora, symposia and discussion groups at various conferences, as well as several special issues of research journals, most recently the *Mathematics Education Research Journal* (Mitchelmore & White, 2007).

One of the approaches to research on abstraction presented on these occasions is *abstraction in context,* or AiC (Hershkowitz, Schwarz, & Dreyfus, 2001). This approach considers abstraction as a process of emergence of knowledge constructs that are new to the learner. In order to describe such processes at a fine-grained level, abstraction in context makes use of a model, the RBC model, which is based on three epistemic actions to be described below. The RBC model has been used for this purpose by different research teams with students of different ages learning about different mathematical topics (including square roots, algebra, probability, rate of change, function transformations, and dynamical systems), in a variety of social and learning contexts (see e.g., Hershkowitz, Hadas, Dreyfus & Schwarz, 2007, and references therein).

In particular, when analysing a solitary learner's construction of a justification for bifurcations in dynamical systems, Dreyfus and Kidron (2006) found an overarching constructing action, within which four secondary constructing actions were nested. These secondary constructing actions were not linearly ordered but went on in parallel and interacted. Interactions included branching of a constructing action from an ongoing one, combining or recombining of constructing actions, and interruption and resumption of constructing actions. The aim of the present paper is to exhibit a facet of the analytic power of the RBC model for abstraction in context, by building on the research by Dreyfus and Kidron, and showing that combining constructing actions indicate crucial steps in the justification process, which lead to an enlightenment of the learner.

Abstraction in Context

Freudenthal has brought forward some of the most important insights to mathematics education in general, and to mathematical abstraction in particular, and this has led his collaborators to the idea of "vertical mathematization" (Treffers & Goffree, 1985). Vertical mathematization points to a process of constructing by learners that typically consists of the reorganization of previous mathematical constructs within mathematics and by mathematical means. This process interweaves previous constructs and leads to a new construct.

AiC adopts this view and defines abstraction as a process of vertically reorganizing previous mathematical constructs within mathematics and by mathematical means so as to lead to a construct that is new to the learner. The genesis of an abstraction passes through a three stage process, which includes the arising of the need for a new construct, the emergence of the new construct, and its consolidation. The need may arise from the design of a learning activity, from the student's interest in the topic or problem under consideration, or from a combination of both; without such a need, however, no process of abstraction will be initiated.

We note that this view of abstraction follows van Oers (2001) in negating the role of decontextualisation in abstraction, and embraces Davydov's dialectic approach (1990) in that it proceeds from an initial unrefined first form to a final coherent construct in a dialectic two way relationship between the concrete and the abstract (see Hershkowitz et al., 2001; Ozmantar & Monaghan, 2007).

Proceedings of the 31st Annual Conference of the Mathematics Education Research Group of Australasia
M. Goos, R. Brown, & K. Makar (Eds.), © MERGA Inc. 2008

Furthermore, we found that activity theory (Leont'ev, 1981) proposes an adequate framework to consider processes that are fundamentally cognitive while taking into account the mathematical, historical, social and learning contexts in which these processes occur. In this, we follow Giest (2005), who considers activity theory as a theoretical basis, which has an underlying constructivist philosophy but allows avoiding a number of problems presented by constructivism.

According to activity theory, outcomes of previous activities naturally turn to artefacts in further ones, a feature which is crucial to trace the genesis and the development of abstraction throughout a succession of activities. The kinds of actions that are relevant to abstraction are *epistemic actions* – actions that pertain to the knowing of the participants and that are observable by participants and researchers. Pontecorvo and Girardet (1993) have used this term to describe how children developed their knowledge on a historical issue during a discussion. The observability is crucial since other participants (teacher or peers) may challenge, share or construct on what is made public.

The RBC model

For the above reasons, Hershkowitz et al. (2001) have chosen to use epistemic actions in order to model the central second stage of the process of abstraction. The three epistemic actions they have found relevant and useful for their purposes are recognizing (R), building with (B) and constructing (C). *Recognizing* takes place when the learner recognizes that a specific previous knowledge construct is relevant to the problem he or she is dealing with. *Building with* is an action comprising the combination of recognized constructs, in order to achieve a localized goal, such as the actualization of a strategy or a justification or the solution of a problem. The model suggests constructing as the central epistemic action of mathematical abstraction. *Constructing* consists of assembling and integrating previous constructs by vertical mathematization to produce a new construct. It refers to the first time the new construct is expressed by the learner either through verbalization or through action. In the case of action, the learner may but need not be fully aware of the new construct. Constructing does not refer to the construct becoming freely and flexibly available to the learner. Becoming freely and flexibly available pertains to the third stage of the genesis of an abstraction, consolidation. Examples for constructing actions will be given below, in the subsection entitled "Combining Constructions".

The RBC model constitutes a methodological tool used for realizing the ideas of abstraction in context. In this sense, it has a somewhat technical nature that serves to identify learner actions at the micro-level. On the other hand, the model also has a definite theoretical significance; Hershkowitz (2007) has discussed the theoretical aspects of the model, its tool aspects, and the relationships between them.

Combining Constructions and Enlightenment

In this section, we use the RBC model, and in particular the notion of constructing action, in order to describe and analyse the knowledge construction process of one mathematician, to be called L, learning about bifurcations in a logistic dynamical system. We have presented the description of this process elsewhere (Dreyfus & Kidron, 2006). While we were acutely aware that the core of the process is a justification, we did not pay attention to the question what justification means for L, nor did we analyse the relationship of this meaning of justification to the constructing actions and the interactions between them. This is the focus of the present paper.

Methodological Considerations

Gathering data about learning processes is methodologically non-trivial. Gathering data about the learning process of a solitary learner presents even greater challenges because there is usually no need for the learner to report about her learning. In the present study, L's epistemic actions were inferred from the detailed notes she took, her Mathematica files, and her computer printouts. Like many mathematicians, L wrote, graphed, drew and sketched a lot, some by hand and some by computer. As is her habit, she carefully dated and kept these notes as well as all computer files and printouts. These documents later served as a window into her thinking for the researchers.

A priori, L's collection of her notes, files and printouts had nothing to do with a plan for the present or any other research. In fact, the idea of using them as raw data for a research study has only been conceived several months after they had been collected. The researchers then constructed a report of the learning process, following an elaborate procedure of several cycles of description by the first author and challenges by the second author. The accuracy of the report may be verified by observing its close correspondence with the raw data, some of which have been published (Dreyfus & Kidron, 2006).

Once the report was agreed upon, we adopted the RBC methodology for identifying epistemic actions (Hershkowitz et al., 2001). The report of the learning experience was divided into episodes, numbered 1-16. Each episode forms a cognitively coherent unit. For the purpose of analysis, each episode was further divided into subunits, called events, and denoted by Latin letters a, b, c, and so on. Events are the equivalent to utterances for the case of a solitary learner; they form the minimal units that can be categorized as epistemic actions according to the RBC model.

One of the tasks of the researchers when using the RBC model is to decide which constructing actions to focus on. For this purpose, each of us independently proposed constructs, which we saw as emerging during the learning process, as candidates for constructing (C) actions. Agreement between us was fairly high, once we agreed on grain size. We identified one overarching C action and four regular C actions whose relevance was obvious to both of us (for more detailed methodological considerations, see Dreyfus and Kidron, 2006). Next, we independently marked each episode according to which of the four regular C actions were active in the episode. There was full agreement between us concerning active C actions. The analysis of the resulting web of interweaving C actions forms the topic of the next subsection.

Combining Constructions

L was interested the following iterative process: Given the quadratic function $f(x) = x + r x (1x)$, where r is a real parameter, consider the sequence of values $\{x_n\}$ produced from an initial value x_0, $0< x_0 <1$, by successive application of f, that is $x_{n+1} = f(x_n)=f^n(x_0)$, for all $n \geq 0$. L discovered empirically that for certain values of r, the sequence of values $\{x_n\}$ converges to a fix point; for somewhat larger values of r, it approaches a process of period 2, for even larger values of r, a process of period 4, and so on. With some support from books and internet sources, she soon computed that the transition from the fix point regime to the 2-periodic regime occurs at r=2 and that this can be computed on the basis of a quadratic polynomial with parameter r, by showing that r=2 is the smallest value for which the discriminant of this polynomial vanishes, and thus the polynomial has a double root.

In the episodes of interest in the present research, L set out to understand where and why the transition from the 2-periodic to the 4-periodic regime occurs. The corresponding constructing actions and some of their interactions are described in this subsection and illustrated by means of the diagram in Figure 1. The time axis of the figure runs from top to bottom.

L spent a considerable number of hours, spread over about two weeks, investigating this question. While the question is analogous to the one concerning the previous transition from fix point to 2-period, the polynomial $p_r(x)$=0 of interest is now of order 12. Web resources led L to the notion of discriminant for a general polynomial (episode 5), which she used with the help of Mathematica (episode 6) to find the numerical value $r = \sqrt{6}$ for the transition point to the 4period. This value of r neatly corresponded to the empirical evidence she had collected. Encouraged by this numerical success, she began to search for the mathematical reasons behind it (episode 7).

One of L's constructing actions, denoted C_1, is the process of finding the four solutions of the polynomial equation $p_r(x)$=0 in the case of period 4. The solution process is considered algebraically and numerically. The focus is on the solutions for fixed values of the parameter r and on relationships between the solutions for different values of r.

Another constructing action, denoted C_2, is the process of constructing algebraic connections between the transition point from the 2-periodic to the 4-periodic regime, the existence of multiple roots of the equation $p_r(x)$=0, and the zeros of the discriminant of $p_r(x)$.

As can be seen in the diagram in Figure 1, at the beginning of episode 7, construction C_1 branches off from the ongoing construction C_2. This happened when L attempted to algebraically connect between the zeros of the discriminant and the transition point, but saw no way to reach this goal because of the complexity of the equations involved. This led her to take a more familiar approach, using numerical calculations belonging to C_1 that she expected to eventually lead to the same goal.

This branching of C_1 from the ongoing C_2 can be explained by means of a refinement of the classification of building-with (B) epistemic actions. Specifically, a class of B-actions was introduced whose purpose it is to organize the problem space so as to make its further investigation possible. Such actions can lead to the requirement of additional constructions and thus branching.

It has been shown how interruptions, resumptions and combining of constructions can be similarly explained by means of refined and/or modified R- and B-epistemic actions. The reader is referred to Dreyfus and Kidron (2006) for details. Here, we focus on combining constructions, such as the combining of C_1 and C_4 in episode 10.

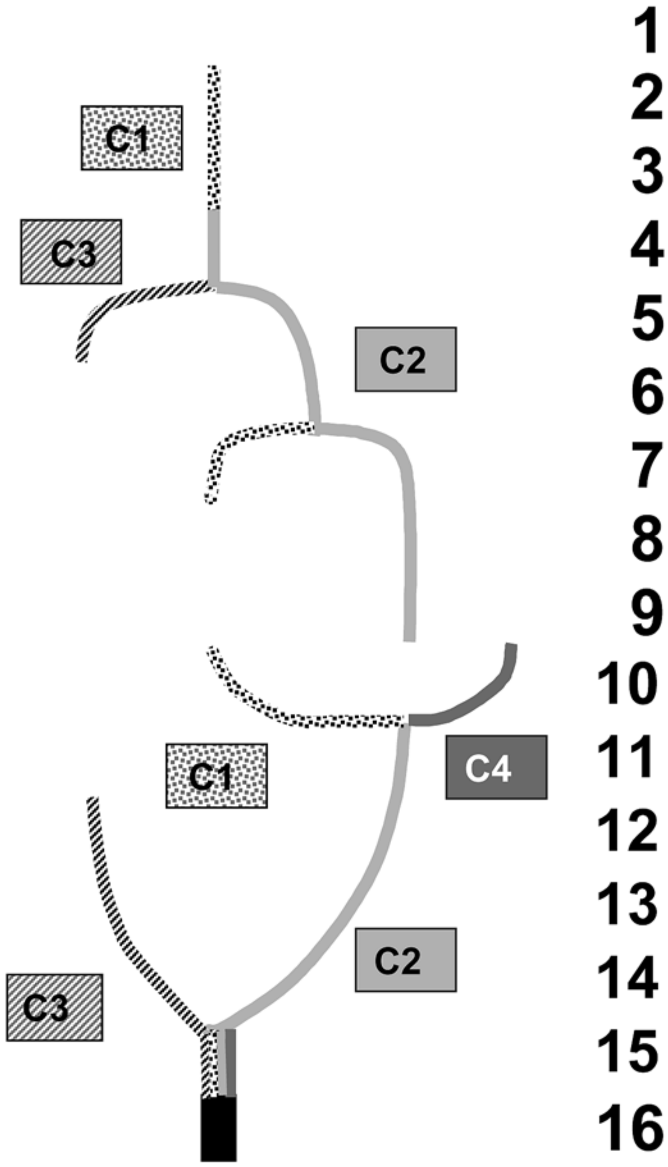

Figure 1. L's interacting parallel constructions.

C_4 denotes the construction of a dynamic view of the bifurcation in which the final state values of the process (the solutions of $p_r(x)=0$) are considered as functions of r. This construction started for L when she had exhausted all her algebraic and numerical resources at the end of episode 9. She considered various graphic representations and focused on the transition of interest:

10d Looking at these values in the bifurcation diagram, my attention was focused on the transition from the 2period to the 4period. This focus was different from the one I had had previously when each time series plot gave a partial picture corresponding to a specific value of the parameter *r*.

10f I looked at the fork-like shape and associated its splitting with the fact that the discriminant vanishes. Suddenly, the bifurcation diagram seemed different, endowed with a new meaning. I looked at it and I could not understand how I failed to see it this way before.

The gradual approach between C_1 and C_4 in Figure 1 expresses the connection in L's thinking of the numerical mode and the graphical mode, which during this approach changed from being static to being dynamic. L's integration of the two modes of thinking results in her view of the transition as a dynamic graphic – numerical process: Her conception of the nature of the parameter r changed from being discrete to being continuous. The two constructions C_1 and C_4 have combined.

While this was intuitively satisfying, it only constituted a first stage since it did not provide the justification L was looking for. She had no algebraic handle on the discriminant, which would connect its zeros to the transition point. She returned to the algebraic mode of thinking C_2, which was now strengthened by her graphic-numerical insight, but she made little progress until, in episode 13, she resumed construction C_3, the process of linking between the derivative of a polynomial, in this case the derivative of $p_r(x)$, the zeros of its discriminant, and the stability of fix points and periods.

This was of interest to L because she vaguely remembered that fix points are stable if the derivative of *f* is smaller than 1 and unstable if the derivative of *f* is bigger than 1. Thus 1 is the limiting value of stability, and at this value a transition occurs.

L carried out a straightforward but rather technical computation showing that $p_r'(x)=0$ (and thus $p_r(x)$ has a multiple root and its discriminant equals zero) if and only if the derivative of $f^4(x)=f(f(f(f(x))))$ equals 1. Possibly without being fully aware, she extended her vague knowledge that the value of 1 of the derivative of f is the limiting value of stability for fix points to the same value 1 of the derivative of $f^4(x)$ being the limiting value of stability for a 4-period.

13i At this moment, I connected the last equality with my previous vague knowledge that fix points change stability when the (absolute value of the) derivative moves across the border 1 as the parameter *r* varies.

13j At last, I found some connection between the fact that there exists a multiple root (therefore, the discriminant equals 0) and the way fix points change stability.

It turned out that this value of 1 of the derivative contained an unwanted minus sign, and L needed more time and effort to clear this up; this is the reason why in Figure 1 the combining of constructions C_3 and C_2 in episode 13 is only partial.

Justification and Enlightenment

In the episodes of interest in the present research, L set out to understand where and why the transition from the 2-periodic to the 4-periodic regime of the logistic dynamical system occurs. At the beginning of episode 8, and again at the beginning of episode 11, right after the first combining of constructions, she expressed what she was looking for, and it was not a formal proof:

8a My aim was to justify why the transition from 2-period to 4-period occurs for the smallest positive real number for which the discriminant equals zero.

8b I felt the need to explain why the requirement that the discriminant equals zero permitted to find the value of the parameter *r* for which the 4-period begins.

11a I was interested in a mathematical explanation why the transition point from 2-period to 4-period is obtained by setting the discriminant equal to zero.

Her question thus was how the value of the discriminant $D = 0$ was connected to the transition between regimes of the dynamical system. Her aim was to convince herself, not others. She felt the need to explain because she wanted to gain more insight.

L's drive to understand the transition was to some extent satisfied intuitively and visually in episode 10, through the combining of the C_1 and C_4 epistemic actions, and she expressed this as follows:

> 10g Now, it seemed to me intuitively clear that at the bifurcation points there must be double solutions and therefore the discriminant should equal zero.

She similarly expressed added insight at the end of episode 13, and again at the end of episode 16:

> 13k Now I understood why at the bifurcation points the discriminant should equal zero. The different elements fit together nicely, like in a puzzle.

> 16d Now I was sure that my mathematical construction will not collapse any more … I was absolutely confident in my justification why the discriminant equals zero, even if the fact that $f^4{}'(x)=1$ was demonstrated at this stage only numerically (based on intuition and authority).

In this sense, L's use of the word justification was very close to Rota's view of enlightenment in the sense of insight into the connections underlying the statement to be justified:

> Verification alone does not give us a clue to the role of a statement within the theory; it does not explain the relevance of the statement … the logical truth of a statement does not enlighten us as to the sense of the statement. … every teacher of mathematics knows that students will not learn by merely grasping the formal truth of a statement. Students must be given some enlightenment as to the sense of the statement. (Rota, 1997, pp. 131-132)

and

> Mathematical proof does not admit degrees. A sequence of steps in an argument is either a proof, or it is meaningless. Heuristic arguments are a common occurrence in the practice of mathematics. However, heuristic arguments do not belong to formal logic … . Proofs given by physicists admit degrees. In physics, two proofs of the same assertion have different degrees of correctness … . A great many characteristics of mathematical thinking are neglected in the formal notion of proof. (ibid., pp. 134-135)

Thus the combining of the C_1 and C_4 actions is an expression of L reaching a first degree of enlightenment, and a feeling of having to some extent explained and justified the structure of the double solution at the transition by means of a dynamic view of the bifurcation. Similarly, the combining of the algebraic mode of thinking of C_2 with the analytic mode of thinking of C_3 in episode 13 expresses L's second degree of enlightenment, which is reinforced by the fact that her previous, if vague, knowledge about the stability of dynamical systems directly confirmed the connection she had established computationally.

Finally, she reached a third degree of enlightenment in episode 16, when she was able to numerically link the dynamic view of the transition – the C_1/C_4 construct from episode 10 – to the link between stability and derivatives – the C_2/C_3 construct from episode 13 – and thus achieve an integration of all four constructing actions.

Conclusion

We remind the reader that the interacting parallel constructions diagram in Figure 1 was obtained by considering L's process of justification as a process of abstraction in context and analysing it by means of the epistemic actions of the RBC model. Only after this description of the process was complete, did we realize the particular meaning, which L associated with the notion of justification, and discover that each additional degree of enlightenment occurs with a combination of two constructions, and each combination of two constructions indicated an additional degree of enlightenment. This enriches the analytic power of the RBC-model: It allows researchers to use the epistemic actions of the RBC model in order to identify a learner's enlightening justification.

References

Davydov, V. V. (1990). *Soviet studies in mathematics education: Vol. 2. Types of generalization in instruction: Logical and psychological problems in the structuring of school curricula* (J. Kilpatrick, Ed., J. Teller, Trans.). Reston, VA: National Council of Teachers of Mathematics. (Original work published in 1972).

Dreyfus, T., & Kidron, I. (2006). Interacting parallel constructions: A solitary student and the bifurcation diagram. *Recherches en didactique des mathématiques, 26*(3), 295-336.

Giest, H. (2005). Zum Verhältnis von Konstruktivismus und Tätigkeitsansatz in der Pädagogik. In F. Radis, M.-L. Braunsteiner, & K. Klement (Eds.), *Badener VorDrucke* (pp. 43-64). Baden A.: Kompetenzzentrum für Forschung und Entwicklung (Schriftenreihe zur Bildungsforschung - Band 3).

Hershkowitz, R. (2007). Contour lines between a model as a theoretical framework and the same model as methodological tool. In B. B. Schwarz & T. Dreyfus (Eds.), *Guided construction of knowledge in classrooms* (Proceedings of an Israel Science Foundation workshop, pp. 39-49). Jerusalem, Israel: The authors.

Hershkowitz, R., Hadas, N., Dreyfus, T., & Schwarz, B. B. (2007). Abstracting processes, from individuals' constructing of knowledge to a group's 'shared knowledge'. *Mathematics Education Research Journal, 19*(2), 41-68.

Hershkowitz, R., Schwarz, B. B., & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education, 32*(2), 195-222.

Leont'ev, A. N. (1981). The problem of activity in psychology. In J. V. Wertsch (Ed.), *The Concept of Activity in Soviet Psychology* (pp. 37-71). Armonk, NY: Sharpe.

Mitchelmore, M., & White, P. (2007). Abstraction in mathematics learning. *Mathematics Education Research Journal, 19*(2), 1-9.

Ozmantar, M.F., & Monaghan, J. (2007). A dialectical approach to the formation of mathematical abstractions. *Mathematics Education Research Journal, 19*(2), 89-112.

Pontecorvo, C., & Girardet, H. (1993). Arguing and reasoning in understanding historical topics. *Cognition and Instruction, 11*(3&4), 365-395.

Rota, G.-C. (1997). *Indiscrete thoughts*. Basel: Birkhäuser.

Treffers, A. & Goffree, F. (1985). Rational analysis of realistic mathematics education – The Wiskobas program. In L. Streefland (Ed.), *Proceedings of the 9th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 97-121). Utrecht, the Netherlands: PME.

Van Oers, B. (2001). Contextualisation for abstraction. *Cognitive Science Quarterly, 1*, 279-305.